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Monotonicity of the Number of Self-Avoiding Walks

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It is shown that the number c_n of self-avoiding walks of length n in $\mathbb{Z}^d$ is an increasing function of n.

KEY WORDS: Self-avoiding walks.

1. INTRODUCTION

A self-avoiding walk (SAW) of length n in $\mathbb{Z}^d$ $(d \geqslant 2)$ is an ordered set $W = (x_0 = 0, x_1, ..., x_n)$ of *distinct* vertices in $\mathbb{Z}^d$ such that

$$|x_k - x_{k-1}| = 1, \qquad k = 1, 2, ..., n \tag{1.1}$$

The idea behind this notion is that a particle starts at 0 and then visits the vertices $x_1, x_2, ..., x_n$ in succession. Thus, it cannot visit a vertex more than once. We can equally well describe W by listing its edges (x_0, x_1), $(x_1, x_2), ..., (x_{n-1}, x_n)$ which together with the vertices form a directed graph.

Let $\mathcal{S}_n$ denote the set of SAWs of length n (we treat d as fixed) and let $c_n = |\mathcal{S}_n|$, the cardinality of $\mathcal{S}_n$. It is difficult to calculate c_n even for fairly moderate n, but a number of bounds and asymptotic results about c_n are known.

Hammersley and Welsh[1] proved that there are constants $\mu > 1$ and $\alpha > 0$ depending on d such that

$$\mu^n \leqslant c_n \leqslant \mu^n \exp(\alpha n^{1/2}) \tag{1.2}$$

Kesten[2] proved that

$$|c_{n+2}/c_n - \mu^2| \leqslant A n^{-1/3} \tag{1.3}$$

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for some constant A. Both papers also contain several related results. There has been little mathematically rigorous progress since Kesten's paper. An interesting exception is the paper by Slade.[4] One obvious question is whether (1.3) can be extended to a result of the form

$$c_{n+1}/c_n \to \mu \qquad \text{(at some specified rate)} \tag{1.4}$$

Many authors have worked on Monte Carlo and related computer-intensive methods to estimate μ and other important quantities arising in connection with SAWs. An extensive list of papers of this type is given by Madras and Sokal.[3] The interest in these results is related to the idea that SAWs can be used as models for linear polymer molecules, as discussed in the review paper by Whittington[5] for example. The large number of papers reflects the fact that the structure of SAWs is so complex and c_n grows so fast that even computer studies are difficult.

Using (1.2), (1.3), and the elementary inequalities $c_n \geqslant d^n$ and $c_{n+1} \leqslant (2d-1)\, c_n$, we deduce that

$$c_n \leqslant \tfrac{4}{3}\mu^{-2} c_{n+2} \leqslant \tfrac{4}{3} d^{-2}(2d-1)\, c_{n+1} \leqslant c_{n+1} \tag{1.5}$$

for n sufficiently large. Also, it is easy to show that $c_n \leqslant c_{n+2}$ for all n. Remarkably, it has not been shown that $c_n \leqslant c_{n+1}$ holds for all n. The main result of this note fills this gap. It is to be hoped that the methodology used in the proof of the following theorem can be modified to help obtain other results about SAWs.

Theorem. For $d \geqslant 2$ and all n, $c_n \leqslant c_{n+1}$.

The basic idea of the proof is to construct a one-to-one function Φ from $\mathscr{S}_n \to \mathscr{S}_{n+1}$. The part of the proof which does not depend on d is given in Section 2. The more complex second part is given in Section 3 for the planar case $d=2$, and in Section 4 for the higher-dimensional case $d>2$.

The following notational conventions are useful. A *segment* $(y_1, y_2,..., y_k)$ of a SAW W is the collection $\{(y_1, y_2), (y_2, y_3),..., (y_{k-1}, y_k)\}$ of successive edges of W. For many purposes the directed nature of the graph W is unimportant. It is often convenient to write "the edge $[y_1, y_2]$" as shorthand for the passage "(y_1, y_2) or (y_2, y_1), whichever is an edge of W." The expression "the segment $[y_1,..., y_k]$" has a similar meaning. If a statement relates to an endpoint of W by mentioning x_1 or x_{n-1}, say, then the order does matter and is written with more care.

2. PROOF OF THE THEOREM: FIRST PART

The *unit vectors* in $\mathbb{Z}^d$ are the $2d$ vectors of the form $(\xi_1,..., \xi_d)$ such that one of the ξ_j is 1 or -1 and the others are all 0. We denote them by $u_1, u_2,..., u_{2d}$, where u_i has a 1 in the ith component for $i \leqslant d$ and a -1 in the $(i-d)$th component for $i > d$. In particular, $u_i + u_{i+d} = 0$ for $1 \leqslant i \leqslant d$. Let $U = \{u_1,..., u_{2d}\}$.

A *rectangle* in $\mathbb{Z}^d$ is a set

$$R := \prod_{j=1}^{d} \{a_j, a_j+1,..., b_j\} \tag{2.1}$$

where a_j and b_j are integers with $a_j \leqslant b_j$. The *faces* of such a rectangle are the $2d$ sets

$$F_i := \prod_{j=1}^{d} H_j, \qquad i = 1, 2,..., 2d \tag{2.2}$$

where $H_j = \{a_j, a_j+1,..., b_j\}$ for $j \notin \{i, i-d\}$, and $H_i = \{b_i\}$ if $i \leqslant d$ and $H_{i-d} = \{a_i\}$ if $i > d$. Suppose $a_i < b_i$ for all $i \leqslant d$; then u_i is orthogonal to F_i for all i and in fact points from F_i toward the outside of R; we call u_i the *outward normal* (*unit*) *vector* for F_i and u_{i+d} or u_{i-d}, as appropriate, the *inward normal* (*unit*) *vector*. All the other u_j in U are parallel to F_i.

For each $i \leqslant 2d$ we define a partial order $\leqslant_i$ on $\mathbb{Z}^d$ as follows: for $i \leqslant d$, $(\xi_1,..., \xi_d) \leqslant_i (\eta_1,..., \eta_d)$ iff $\xi_j \leqslant \eta_j$ for all j; and for $d < i \leqslant 2d$, $(\xi_1,..., \xi_d) \leqslant_i (\eta_1,..., \eta_d)$ iff $\xi_j \leqslant \eta_j$ for all $j \neq i-d$ and $\xi_{i-d} \geqslant \eta_{i-d}$. A unit vector u is *upward* (*downward*) with respect to $\leqslant_i$ if $0 \leqslant_i u$ (respectively $u \leqslant_i 0$).

Now let $W = (0, x_1,..., x_n) \in \mathscr{S}_n$. The *adjacent* vertices (in W) to x_k are x_{k-1} and x_{k+1} (or only one of these if $k = 0$ or $k = n$). We let R be the least rectangle containing W. There is at least one vertex of W in each face F_i of R. The vertex $(\xi_1,..., \xi_d)$ of W which is lexicographically largest among all vertices in $W \cap F_i$ is called the *pivot point* of W for F_i. It is clear that the pivot point for F_i is a maximal vertex relative to $\leqslant_i$ among all vertices of W. Note that the pivot point *for* F_i is *in* F_i, but may also be *in* F_j for some $j \neq i$. We will define $W' := (0, v_1,..., v_{n+1}) = \Phi(W)$ in a manner which depends on which of the following three disjoint subclasses of $\mathscr{S}_n$ contains W:

$$\mathscr{J}_1 := \{W \in \mathscr{S}_n\text{: for some } i\text{, either } x_n \in F_i \text{ or } x_n + u_i \in F_i \backslash W\}$$

$$\mathscr{J}_2 := \{W \in \mathscr{S}_n \backslash \mathscr{J}_1\text{: for some } i\text{, either } 0 \in F_i \text{ or } u_i \in F_i \backslash W\}$$

$$\mathscr{J}_3 := \mathscr{S}_n \backslash (\mathscr{J}_1 \cup \mathscr{J}_2)$$

If $W \in \mathscr{J}_1$, choose i such that $x_n \in F_i$ or $x_n + u_i \in F_i \backslash W$ and define $W' = (0, x_1,..., x_n, x_n + u_i)$. If $W \in \mathscr{J}_2$, choose i such that $0 \in F_i$ or $u_i \in F_i \backslash W$ and define $W' = (0, -u_i, x_1 - u_i, \ldots, x_n - u_i)$. In words, W' is formed from W by appending a vertex before 0 and then translating the graph to place the new vertex at 0. If $W \in \mathscr{J}_3$, we construct W' by first perturbing W near a pivot point (details to be given later) to obtain a SAW $W'' = (0, v_1, \ldots, v_{n+2}) \in \mathscr{S}_{n+2}$ and then deleting the last vertex v_{n+2} and edge from W''.

Let R' be the least rectangle containing W', and let F_i', $i \leqslant 2d$, be its faces and p_i', $i \leqslant 2d$, be the pivot point of W' for F_i'. Although we have not yet given the details of the perturbation for $W \in \mathscr{J}_3$, we temporarily take it for granted that the following conditions hold for $W \in \mathscr{J}_3$:

$$R \subset R' \tag{2.3}$$

$$x_n \notin W' \tag{2.4}$$

and

$$v_{n+1} = x_{n-1} \in W \tag{2.5}$$

As a first step toward showing that Φ is one-to-one, we have the following result.

Lemma 1. Given $W' = \Phi(W)$ for some $W \in \mathscr{S}_n$, we can determine which of $\mathscr{J}_1$, $\mathscr{J}_2$, and $\mathscr{J}_3$ contains W.

Proof. If $W \in \mathscr{J}_1$, then, for some i,

$$v_{n+1} \in F_i' \qquad \text{and} \qquad v_n \notin F_i' \tag{2.6}$$

If $W \in \mathscr{J}_2$, then there is no i for which (2.6) holds, but, for some i,

$$0 \in F_i' \qquad \text{and} \qquad v_1 \notin F_i' \tag{2.7}$$

Now suppose $W \in \mathscr{J}_3$. If $x_{n-1} = v_{n+1} \in F_i'$ for some i (necessarily unique), then $x_n - x_{n-1} = -u_i$. By (2.3) and (2.4), $v_n - v_{n+1} \neq -u_i$, so $v_n \in F_i'$ also, contrary to (2.6). Thus, (2.6) holds if and only if $W \in \mathscr{J}_1$. For $W \in \mathscr{J}_3$, it is also clear that $0 \notin F_i'$ by (2.3) and the definition of $\mathscr{J}_2$. Thus, if (2.6) fails, then (2.7) holds if and only if $W \in \mathscr{J}_2$. ∎

If $W \in \mathscr{J}_1$ ($W \in \mathscr{J}_2$), one can recover W from W' by deleting the final vertex and edge (the first vertex and edge and translating by $-v_1$, respectively). Our remaining task is to define Φ on $\mathscr{J}_3$ in such a way that this recovery process can be performed on $\mathscr{J}_3$ also.

The details of the definition of Φ on $\mathscr{J}_3$ are somewhat different in the cases $d = 2$ and $d > 2$. The basic idea is the same, however, and it is useful to prepare for the details by giving an outline of the construction. We assume henceforth that $W \in \mathscr{J}_3$.

A pivot point p_i of W is said to be an *upwardly mobile pivot point* (UMPP) for F_i if W satisfies certain conditions near p_i. If p_i is a UMPP, then there are at least $2d-1$ possible perturbations (in an upward direction relative to $\leqslant_i$) near p_i such that the pivot point p_i' of the perturbed SAW is also a UMPP for F_i'. If p_i is not upwardly mobile, there is at least one possible such perturbation. Then W' is constructed from W by choosing a different perturbation near some p_i according to the direction of the edge (x_{n-1}, x_n) to get W'' and then deleting that edge from W'' to get W'. If the choice of perturbation is made judiciously, its location can be recovered by locating the largest i (in some ordering) for which p_i' is a UMPP, and x_n and W can then be recovered by determining exactly which perturbation was used.

3. COMPLETION OF THE PROOF IN THE PLANAR CASE

When $d=2$, we can think of $F_1,..., F_4$ as the right, top, left, and bottom faces, respectively, of R. The pivot points are the rightmost or top vertices of W for their faces. Since $W \in \mathscr{J}_3$, $[p_4 - r_0 u_1 + u_2, p_4 - r_0 u_1, p_4 - (r_0 - 1) u_1,..., p_4, p_4 + u_2]$ is a segment of W for some $r_0 > 0$.

Definition. We call p_4 a UMPP for F_4 if $W \cap \{p_4 + u_2 + ru_1: r > 0\} = \varnothing$ and one of the following five disjoint conditions also holds (see Fig. 1):

(A) $r_0 > 1$.

(B) $r_0 = 1$, $[p_4 - 2u_1 + u_2, p_4 - u_1 + u_2]$, and $[p_4 + u_2, p_4 + 2u_2]$ are edges of W and

$$W \cap \{p_4 - ru_1: r > 1\} = W \cap \{p_4 + 2u_2 + ru_1: r > 0\} = \varnothing$$

(C) $r_0 = 1$ and $W \cap \{p_4 + u_2 - ru_1: r > 1\} = \varnothing$.

(D) $r_0 = 1$, (B) fails, $W \cap \{p_4 - ru_1: r > 1\} = \varnothing$, and $p_4 + u_2 - ru_1 \in W$ for some $r > 2$.

(E) $r_0 = 1$ and for some $r_1 > 1$, $p_4 - r_1 u_1 \in W$, and $W \cap \{p_4 + u_2 - ru_1: r > r_1\} = \varnothing$.

The pivot point p_1 for F_1 is said to be a UMPP if $W \cap \{p_1 - u_1 + ru_2: r > 0\} = \varnothing$.

The pivot points p_4' and p_1' are called UMPPs if the corresponding statements hold for W'.

The Three Perturbations. We next construct three SAWs in $\mathscr{S}_{n+2}$ by perturbing W near one of the pivot points p_4, p_1, or p_2. There are three cases.

Case 1. p_4 is a UMPP for F_4. The three perturbations are:

(i) Replace the segment $[p_4 - r_0 u_1, ..., p_4]$ by the segment

$[p_4 - r_0 u_1,\ p_4 - r_0 u_1 - u_2,\ p_4 - (r_0 - 1) u_1 - u_2, ..., p_4 - u_2,\ p_4]$

(ii) Replace $[p_4, p_4 + u_2]$ by

$$[p_4,\ p_4 + u_1,\ p_4 + u_2 + u_1,\ p_4 + u_2]$$

(iii) If (A) holds, replace $[p_4 - u_1, p_4]$ by

$$[p_4 - u_1,\ p_4 - u_2 - u_1,\ p_4 - u_2,\ p_4]$$

If (B) holds, replace $[p_4, p_4 + u_2, p_4 + 2u_2]$ by

$$[p_4,\ p_4 + u_1,\ p_4 + u_1 + u_2,\ p_4 + u_1 + 2u_2,\ p_4 + 2u_2]$$

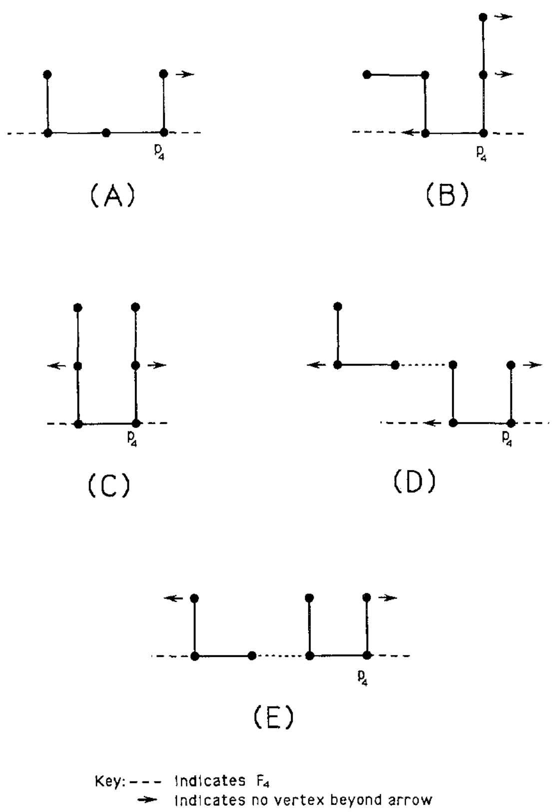

Fig. 1. Upwardly mobile pivot points for F_4.

If (C) holds, replace $[p_4 - u_1, p_4 - u_1 + u_2]$ by

$$[p_4 - u_1, p_4 - 2u_1, p_4 - 2u_1 + u_2, p_4 - u_1 + u_2]$$

If (D) holds, and q is the leftmost point of $W \cap \{p_4 + u_2 - ru_1 : r > 2\}$, replace $[q, q + u_1]$ by

$$[q, q - u_2, q + u_1 - u_2, q + u_1]$$

If (E) holds, replace $[p_4 - r_1 u_1, p_4 - r_1 u_1 + u_2]$ by

$$[p_4 - r_1 u_1, p_4 - (r_1 + 1) u_1, p_4 - (r_1 + 1) u_1 + u_2, p_4 - r_1 u_1 + u_2]$$

Case 2. p_4 is not a UMPP for F_4 and p_1 is a UMPP for F_1. The three perturbations are:

(iv) Replace $[p_1, p_1 - u_2]$ by $[p_1, p_1 + u_1, p_1 - u_2 + u_1, p_1 - u_2]$.
(v) Replace $[p_1, p_1 - u_1]$ by $[p_1, p_1 + u_2, p_1 + u_2 - u_1, p_1 - u_1]$.
(vi) Same as (i).

Case 3. p_4 is not a UMPP for F_4, and p_1 is not a UMPP for F_1. The three perturbations are:

(vii) Replace $[p_2, p_2 - u_1]$ by $[p_2, p_2 + u_2, p_2 - u_1 + u_2, p_2 - u_1]$.
(viii) Same as (iv).
(ix) Same as (i).

We note that each of the above perturbations yields a SAW in $\mathscr{S}_{n+2}$.

Definition of $\Phi(W)$ for $W \in \mathscr{J}_3$. Given W, we first note which of the above three cases applies and then construct $W'' \in \mathscr{S}_{n+2}$ by perturbing W in one of the three indicated ways, subject to the following two restrictions: first, if $W \in \mathscr{S}_n$ and $W_1 \in \mathscr{S}_n$ are the same up to but not including their final edges, then different perturbations are applied to W and W_1; and second, if $x_{n-1} \in F_i$ for some necessarily unique i, then a perturbation of type (i), (vi), or (ix) is used, while if p_4 is a UMPP satisfying (B) and $x_n = p_4 + 2u_2$, then a perturbation of type (ii) is used. If the resulting SAW in $\mathscr{S}_{n+2}$ is $W'' = (0, v_1, v_2, \ldots, v_{n+2})$, we define $W' = \Phi(W) = (0, v_1, \ldots, v_{n+1})$.

We note that (2.3) and (2.4) hold in all cases. The second restriction on the choice of perturbation guarantees that the edge (x_{n-1}, x_n) is in W'' and hence that $v_{n+1} = x_{n-1}$. Thus, (2.5) also holds.

The One-to-Oneness of Φ on $\mathscr{J}_3$. Suppose first that the perturbation used to construct W'' from W is of type (i), (ii), (iii), (vi), or (ix). It is tedious but straightforward to see that the pivot point p_4' is a UMPP for F'_4 in this case. For example, if p_4 is a UMPP satisfying (D) for F_4, the perturbation of type (iii) makes p'_4 a UMPP satisfying (E) for F'_4. Next suppose that the perturbation is of type (iv), (v), or (viii). Then p'_1 is a UMPP for F'_1. Also, these perturbations are only used if p_4 is not a UMPP for F_4, which implies in particular that if $p_4 \in F_1$, then $[p_4 - u_1 + u_2, p_4 - u_1, p_4, p_4 + u_2, p_4 + 2u_2]$ is a segment of W. Hence $p'_4 = p_4$ and consequently is not a UMPP for F'_4. Finally, suppose p_4 is not a UMPP for F_4; p_1 is not a UMPP for F_1; and the perturbation is of type (vii). It is clear that $p'_4 = p_4$ is not a UMPP for F'_4 and $p'_1 = p_1$ is not a UMPP for F'_1.

Thus, we can determine from W' which of these three situations applies. An examination of the configurations that result from each type of perturbation shows that they are distinguishable in terms of their nature near p'_4 and p'_1. Hence $W^* = (0, x_1, \ldots, x_{n-1})$ can be recovered from W'. We can then recover W by using the association between the perturbations and the possible edges (x_{n-1}, x_n).

4. COMPLETION OF THE PROOF IN HIGHER DIMENSIONS

Let $W \in \mathscr{J}_3$. When $d > 2$, we can use a much simpler definition of UMPP and a smaller variety of perturbations. As before, let p_i be the pivot point for F_i, $i \leqslant 2d$. We recall that if y is adjacent to p_i, then $y \leqslant_i p_i$.

Definition. We say p_i is a UMPP for F_i if $s \in W$ and $y \leqslant_i s$, where y is adjacent to p_i, together imply that $s = y$ or $s = p_i$.

The $2d-1$ Perturbations. A *type I perturbation* on F_i is the replacement of $[y, p_i]$ by $[y, y+u, p_i+u, p_i]$, where y is adjacent to p_i in W, u is an upward unit vector relative to $\leqslant_i$, and $u \neq p_i - y$. A *type II perturbation* on F_i is the replacement of $[y, p_i, z]$ by $[y, y+u, p_i+u, z+u, z]$, where y and z are adjacent to p_i in W, u is upward relative to $\leqslant_i$, $u \neq p_i - y$, and $u \neq p_i - z$. A *type I* perturbation* on F_i is a type I perturbation on F_i for which $u = u_i$. In all cases, u is called the *perturbing unit vector* and, if W'' is a SAW obtained by perturbing W, the vertices in $W'' \setminus W$ are called the vertices *added to* W.

Since $p_i \in F_i$, at least one of its adjacent vertices is in F_i, so at least one type I* perturbation on F_i is possible in the sense that it gives rise to a SAW in $\mathscr{S}_{n+2}$. If p_i is a UMPP, then all $2(d-1)$ type I and $d-2$ type II perturbations are possible in this sense. Since $d \geqslant 3$, this provides at least $2d-1$ perturbations. (The nonexistence of type II perturbations when $d=2$ is the reason the planar case requires a separate argument.)

As in the planar situation, we choose our $2d-1$ perturbations according to which of three cases applies. If p_{2d} is a UMPP for F_{2d}, choose $2d-2$ type I perturbations and one type II perturbation, all on F_{2d}. If p_{2d} is not a UMPP for F_{2d}, but p_i is a UMPP for F_i for some $i < 2d$, take the largest such i and choose $2d-2$ type I perturbations on F_i and one type I* perturbation on F_{2d}. If no p_i is a UMPP for F_i, choose one type I* perturbation on each of $2d-1$ faces.

Definition of $\Phi(W)$. Given $W \in \mathscr{J}_3$, we define $W'' = (0, v_1,..., v_{n+2})$ and $W' = \Phi(W) = (0, v_1,..., v_{n+1})$ as in the planar case, except that the second restriction becomes: if $x_{n-1} \in F_i$ for some i, then a type I* perturbation is used.

As before, (2.3)–(2.5) hold.

The One-to-Oneness of Φ on $\mathscr{J}_3$

Lemma 2. Suppose that the perturbation used in the construction of W'' is on F_j. Then p'_j is a UMPP for F'_j and $p'_j \notin W$.

Proof. If p_j is a UMPP for F_j, the conclusion is clear, whereas if it is not, then the perturbation is of type I* and the conclusion again follows. ∎

If there were no interaction between faces, we could locate the perturbation by finding the largest j for which F'_j has a UMPP. However, if p_j is near the edge of F_j, a perturbation on F_j may give rise to a UMPP for F'_i where $i > j$ in one of two ways: first, it is possible that $p'_i \neq p_i$, and second, it is possible that $p'_i = p_i$, but that the adjacent vertices to p'_i in W' are different from the adjacent vertices to p_i in W. These complications, which we managed to avoid in the planar case, are dealt with in the following lemmas.

Lemma 3. Let the perturbation be on F_j and let $i \geqslant j$ be such that p'_i $(= v_k$, say) is a UMPP for F'_i and $p'_i \notin W$. Then exactly one of the following three statements holds:

(S_1) $v_k - v_{k-1} = v_{k+1} - v_{k+2}$ and, if $i > d$ and $v_k - v_{k-1} = u_i$, then $F'_i \cap W' = \{v_k, v_{k+1}\}$.

(S_2) $v_k - v_{k+1} = v_{k-1} - v_{k-2}$ and, if $i > d$ and $v_k - v_{k+1} = u_i$, then $F'_i \cap W' = \{v_k, v_{k-1}\}$.

(S_3) $v_{k+2} - v_{k+1} = v_{k-2} - v_{k-1}$ and $v_k + v_{k+2} + v_{k-1} \notin W'$.

The first two statements can only hold for type I perturbations and the third only for type II perturbations. In the three cases, the perturbation replaces $[v_{k-1}, v_{k+2}]$ by $[v_{k-1},..., v_{k+2}]$, $[v_{k-2}, v_{k+1}]$ by $[v_{k-2},..., v_{k+1}]$, and $[v_{k-2}, v_k + v_{k+2} - v_{k+1}, v_{k+2}]$ by $[v_{k-2},..., v_{k+2}]$, respectively.

Proof. If the perturbation is of type I and the added vertices are v_k and v_{k+1}, then the first equation in (S_1) holds. If $i > d$ and $v_k - v_{k-1} = u_i$, then the perturbation is of type I* on F_i and $i=j$, since u_i is upward only with respect to $\leqslant_i$; but then $F_i' \cap W' = \{v_k, v_{k+1}\}$. Statement (S_2) similarly follows if the added vertices are v_k and v_{k-1}. If the perturbation is of type II, which implies $i=j=2d$, then v_k, v_{k-1}, and v_{k+1} are the three added vertices and (S_3) follows. The equations in (S_1) and (S_2) together imply $v_{k+2}=v_{k-2}$, so they cannot both hold. Similarly, (S_3) is inconsistent with either (S_1) or (S_2). The final statement is now obvious. ▮

Lemma 4. Suppose the perturbation is on F_j. Let $i \geqslant j$ be such that $p_i' = v_k$ is a UMPP for F_i' and $p_i' \in W$. Then none of (S_1), (S_2), or (S_3) holds.

Proof. By Lemma 2, $i>j$, so $j<2d$. Therefore, the perturbation is of type I. Since $p_i' \in F_i$, it must in fact be the pivot point for F_i, that is, $p_i' = p_i$. On the other hand, p_i cannot be a UMPP for F_i, for if it were, the perturbation would be on F_i rather than F_j. The only situation consistent with p_i being a UMPP for F_i' but not for F_i is that one of the adjacent vertices of p_i in W', say v_{k+1}, is not in W. The perturbing vector u must then satisfy

$$u = v_{k+1} - v_k = v_{k+2} - v_{k+3} \tag{4.1}$$

and the two added vertices must be v_{k+1} and v_{k+2}. Since (4.1) and (S_1) together imply $v_{k+3} = v_{k-1}$, (S_1) fails. Since $v_{k+3} = v_k + v_{k+2} - v_{k+1}$, (S_3) fails. Note that u must be upward with respect to $\leqslant_j$, but, since v_k is a UMPP for F_i', $u = v_{k+1} - v_k$ must be downward with respect to $\leqslant_i$. Hence $j>d$ and $u=u_j$ or else $i>d$ and $u = -u_i$. Suppose, first, that $u=u_j$ and $j>d$. The perturbation is therefore of type I* and

$$F_j' \cap W' = \{v_{k+1}, v_{k+2}\} \tag{4.2}$$

It follows that $v_k \in F_j \cap F_i$. Since v_k is a pivot point of F_i', we have $v_{k+1} \leqslant_i v_k$ and $v_{k-1} \leqslant_i v_k$. Since $v_k \in F_j$ and $v_{k+1} \notin F_j$, we deduce that $v_{k-1} \in F_j$. If (S_2) held, then $v_{k-2} = v_{k+1} - v_k + v_{k-1} \in F_j'$, contrary to (4.2). So (S_2) must fail in the case $u=u_j$. Now suppose that $u=-u_i$ and $i>d$. By (4.1), we have that $v_{k+3} \in F_i' = F_i$, so (S_2) must fail in this case. ▮

The rest of the proof of the theorem is easy. Pick the largest i such that p_i' is a UMPP for F_i' and one of (S_1), (S_2), or (S_3) holds. By Lemmas 2 and 3 such an i exists and $i \geqslant j$, where the perturbation is on F_j. By Lemma 4, $p_i' \notin W$. Then, by Lemma 3 the perturbation can be identified and all but the final edge of W can be recovered. That edge is then recovered via the association between that edge and the choice of perturbation in the construction of W''. ▮

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